# KALMAN FILTER PERFORMANCE DEGRADATION WITH AN ERRONEOUS PLANT MODEL

LARRY BLAYNE NOFZIGER
and
GERALD LEE DEVINS

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# KALMAN FILTER PERFORMANCE DEGRADATION

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by

Larry Blayne Nofziger
Lieutenant Commander, United States Navy
B.S. in Electrical Engineering
Indiana Institute of Technology, 1954

and

Gerald Lee Devins
Lieutenant, United States Navy
B.S. in Engineering Electronics
Naval Postgraduate School, 1965

[redacted]

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ABSTRACT

This investigation is concerned with the effects of employing a Kalman filter to estimate the states in a system for which the mathematical model is inaccurate. Consideration is given to both intentional and unintentional mis-identification of parameters in the assumed plant dynamics. An algorithm consisting of four matrix equations is derived which yields the actual covariance of estimation error when errors in the assumed model are known. Depending upon the gain sequence used, the derived equations can be used to either 1) produce optimal estimates when errors are deliberate or 2) aid in the determination of mis-identification costs in terms of filter performance degradation if the relative accuracy of parameter identification is known.

Analytic examples of scalar cases are included, as well as computer simulations for specific higher order systems, including the employment of a second order filter model with a fourth order plant.

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# TABLE OF CONTENTS

## LIST OF ILLUSTRATIONS

## LIST OF SYMBOLS AND ABBREVIATIONS

| Symbol | Dimensions | Meaning |
|---|---|---|
| A | n x n | Canonic matrix of state equation coefficients |
| a | scalar | Amplification, model parameter, amplification and feedback coefficient |
| B | n x m | Input distribution matrix |
| D(k) | n x n | $E\{\underline{x}(k)\underline{x}^T(k)\}$ |
| E{x} | operator | Expected value of x |
| $G_o(k)$ | n x p | Optimal weighting for $k^{th}$ observation |
| $G_f(k)$ | n x p | Filter " " " " |
| H | p x n | Observability matrix |
| I | n x n | Identity matrix |
| J | scalar | Performance index, trace of P(k/k) |
| K(k) | n x n | $E\{\underline{\hat{x}}(k/k)\underline{x}^T(k)\}$ |
| k | integer | Index or sequential stage |
| k/k | integers | "at $k^{th}$ iteration given k samples" |
| k+1/k | integers | "predicted at $(k+1)^{th}$ iteration given k samples" |
| m | integer | Number of inputs |
| n | integer | Number of states, order of system |
| $P_a(\ )$ | n x n | Actual covariance matrix (calculated) |
| $P_f(\ )$ | n x n | Filter " " " |
| $P_o(\ )$ | n x n | Optimum " " " |
| $P_e(\ )$ | n x n | Ensemble average covariance of estimation error |
| p | integer | Number of states observed |
| $Q_f$ | n x n | $\Gamma_f E\{\underline{u}(k)\underline{u}^T(k)\}\Gamma_f^T$ |

| | | |
|---|---|---|
| $Q_p$ | n x n | $\Gamma_p E\{\underline{u}(k)\underline{u}^T(k)\}\Gamma_p^T$ |
| R | p x p | Measurement noise covariance matrix |
| $\underline{u}(k)$ | m x 1 | Excitation vector or input vector |
| $\underline{v}(k)$ | p x 1 | Measurement noise vector |
| $\underline{x}(k)$ | n x 1 | State vector at $k^{th}$ sample |
| $\underline{\hat{x}}(k/k)$ | n x 1 | Estimate of state vector given k samples |
| $\underline{\hat{x}}(k+1/k)$ | n x 1 | Predicted value of x(k+1) given k samples |
| $\underline{z}(k)$ | p x 1 | Vector of observations at $k^{th}$ sample |
| $\alpha$ | scalar | General plant parameter |
| $\Gamma_f(T)$ | n x m | Filter transmission matrix |
| $\Gamma_p(T)$ | n x m | Plant " " |
| $\zeta$ | scalar | Model parameter, damping factor |
| $\Phi_f(T)$ | n x n | Filter state transition matrix |
| $\Phi_p(T)$ | n x n | Plant " " " |
| $\Omega$ | m x m | Covariance matrix of input excitation |
| $\omega$ | scalar | Model parameter, natural frequency |

CHAPTER 1

## INTRODUCTION

In recent years, a considerable portion of the literature in the field of automatic control has been concerned with plant identification and state estimation. In most control problems, it is first necessary to establish a suitable mathematical model of the process to be controlled in order to perform any meaningful analysis or synthesis. Then, if some sort of observation of the process is available, this observation along with the mathematical model, and at least a probabilistic description of the forcing function, provides the necessary information to implement an estimation scheme which will give a measure of what the plant is doing at the present time, has done in the past, or will do in the future.

Whenever estimation is attempted with an inaccurate mathematical model, the estimation accuracy must of necessity deteriorate. The investigation reported here is concerned with the degradation of estimation accuracy when an erroneous model of the plant dynamics is employed.

At this point it is necessary to explain two reasons for not using an accurate mathematical model in the estimation scheme. The first is unintentional, a result of the simple fact that the mathematical model which most accurately describes the plant is not known. A second possible reason might be the deliberate employment of a low-order model of a more complicated plant. Because the mathematical involvement of most estimation schemes is

inescapably tied to system order, much computational time can be saved whenever the model order can be reduced. Such reduction may be necessary for "real time" estimation, at the expense of estimation accuracy. This is assuming of course that in the problem at hand, estimates of the higher order states are not needed.

The three "tenses" of estimation mentioned above are known as filtering, smoothing, and prediction, respectively. In current practice "what the plant is doing" is described mathematically by a state vector, the components of which represent the minimum number of entities required to completely describe the condition of the plant. As an example, if the plant were a passive electrical network, the required state vector components could be inductor currents and capacitor voltages.

This investigation was restricted to a discrete-sampled-data description of the plant and estimation scheme. The use of this mathematical framework leads to a sequential filtering scheme. In this technique of estimation, as in most, a weighting is given to each observation according to how much new information it gives relative to that already received. In current practice this weighting or "filter gain" is calculated to minimize (or maximize) some performance index which has previously been defined. When linear operations on the data are employed and the index to be minimized is mean squared estimation error, the resulting estimation scheme is called a Kalman Filter. [1, 5].

The remainder of this Chapter includes the development of the plant model, a brief review of Kalman filter equations and a statement of the problem to be investigated. In Chapter 2, the results of some other recent investigations are discussed. In Chapter 3, a set of recursive equations for finding a measure of estimation degradation is derived, followed by three simple examples. Results of digital computer simulations using the derived equations for several examples are presented in Chapter 4. Chapter 5 consists of comments on the results of the computer simulations.

## THE PLANT MODEL

Mathematical formulation of the problem proceeds from the assumption that there exists a set of linear, constant coefficient, first order differential equations which adequately describe the plant, or message generating process. These are of the form

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{1-1}$$

Where $\underline{x}$ is the state vector of n components for an nth order system, $\underline{u}$ is the m x 1 input vector, and A is n x n, B is n x m, both matrices of constants. The sampled data matrix difference equation which gives response at sampling instants becomes

$$\underline{x}(k+1) = \Phi(T)\underline{x}(k) + \Gamma(T)\underline{u}(k) \tag{1-2}$$

where $\Phi(T)$ is the n x n discrete state transition matrix, $\Gamma(T)$ is the n x m input distribution matrix and $\underline{u}(k)$ is a sampled and zero order held input vector. Constant differential equation coefficients are not necessary, but are used here for simplicity. The magnitude of each component of $\underline{u}(k)$

is assumed to be a normally distributed random variable with zero mean and known variance [5]. The observations of the system states are assumed to be contaminated by additive gaussian white noise of zero mean and known variance. In matrix notation the observation vector $\underline{z}$ at the kth sampling instant is given as

$$\underline{z}(k) = H\underline{x}(k) + \underline{v}(k) \tag{1-3}$$

where H is the p x n observation matrix, here assumed known and constant and $\underline{v}(k)$ is the p x 1 vector of additive measurement noise. A block diagram depicting the above conditions is shown in Figure 1-1. The double lines represent vector signal flow.

## THE KALMAN FILTER

The sequential estimation technique developed by R.E. Kalman and expanded by others, takes the plant description as defined in the preceding section, and produces an estimate $\hat{\underline{x}}(k/k)$ of the state vector $\underline{x}(k)$ at the kth iteration given k observations. This estimation scheme is commonly called the Kalman filter and can be described mathematically as

$$\hat{\underline{x}}(k/k) = \Phi(T)\hat{\underline{x}}(k-1/k-1) + G(k)[\underline{z}(k) - H\Phi(T)\hat{\underline{x}}(k-1/k-1)] \tag{1-4}$$

where G(k) is the filter weighting or gain applied at the $k^{th}$ iteration. This gain is calculated to minimize the scalar performance index

$$J \equiv E\{[\underline{x}(k)-\hat{\underline{x}}(k/k)]^T[x(k)-\hat{\underline{x}}(k/k)]\} \tag{1-5}$$

i.e., the mean square estimation error.

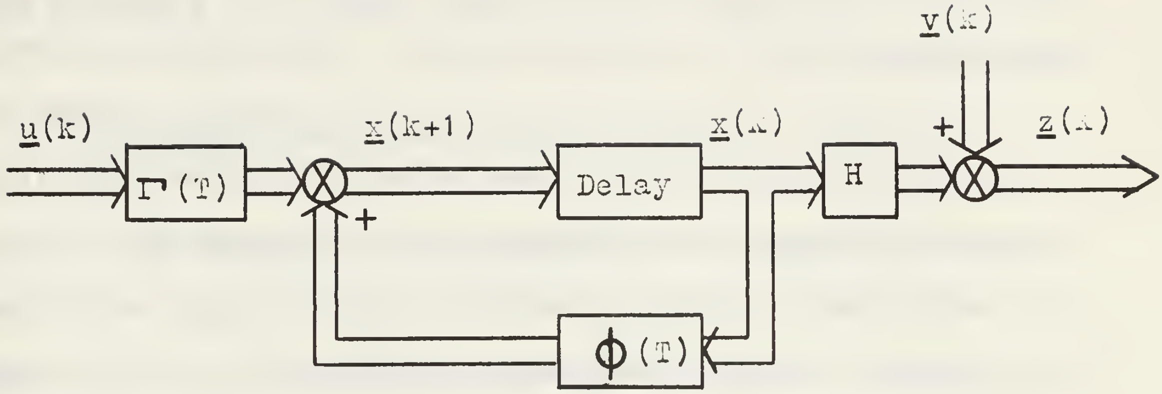


Fig. 1-1 The Plant Model

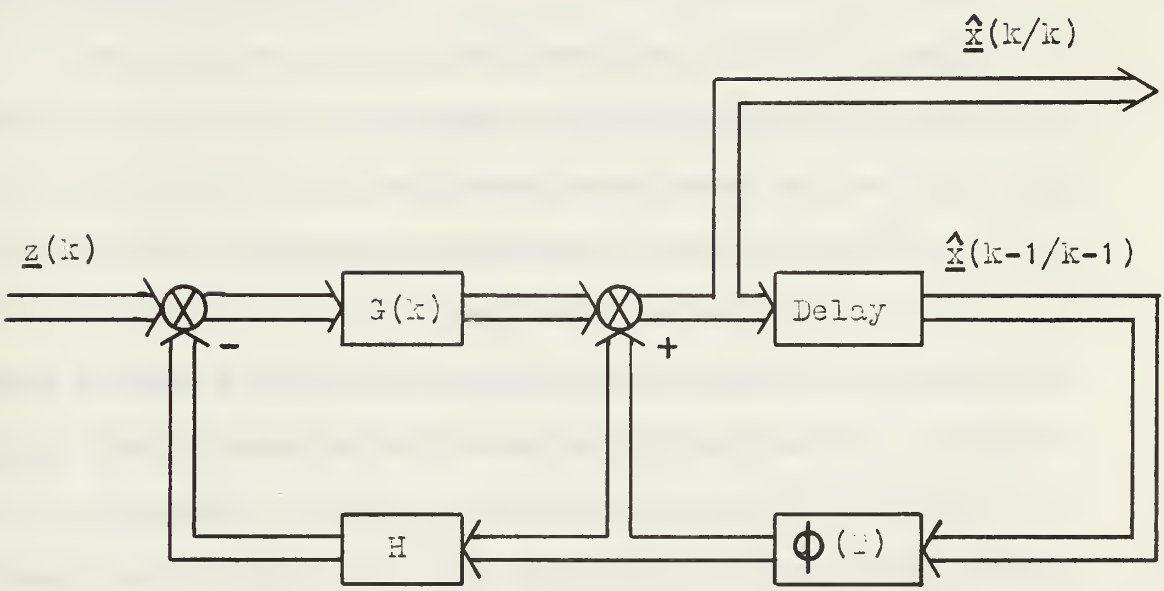


Fig. 1-2 Kalman Filter Operations

The calculation of G(k) is facilitated by defining a matrix of covariances of estimation error as

$$P(k/k) \equiv E\{[\underline{x}(k)-\underline{\hat{x}}(k/k)][\underline{x}(k)-\underline{\hat{x}}(k/k)]^T\} \tag{1-6}$$

The trace of P(k/k) is of course simply J. With the further definition,

$$P(k+1/k) \equiv E\{[\underline{x}(k+1)-\underline{\hat{x}}(k+1/k)][\underline{x}(k+1)-\underline{\hat{x}}(k+1/k)]^T\} \tag{1-7}$$

the Kalman sequential equations can be stated as;

$$P(k+1/k) = \Phi(T)P(k/k)\Phi^T(T) + Q(k) \tag{1-8}$$

$$G(k+1) = P(k+1/k)H^T[HP(k+1/k)H^T+R]^{-1} \tag{1-9}$$

$$\begin{aligned} P(k+1/k+1) = {} & [I-G(k+1)H]P(k+1/k) \\ & - P(k+1/k)H^TG^T(k+1) \\ & + G(k+1)[HP(k+1/k)H^T+R(k)]G^T(k+1) \end{aligned} \tag{1-10}$$

Equation 1-10 can be reduced to

$$P(k+1/k+1) = [I-G(k+1)H]P(k+1/k) \tag{1-11}$$

where

$R(k) \equiv E\{\underline{v}(k)\underline{v}^T(k)\}$

$Q(k) \equiv \Gamma(T)E\{\underline{u}(k)\underline{u}^T(k)\}\Gamma^T(t)$

$E\{\cdot\}$ = the expectation operation

$(\ )^T$ = the transpose operation

$(\ )^{-1}$ = the matrix inversion operation

$I$ = the identity matrix

R(k) is a p x p diagonal matrix based upon a priori knowledge of the average measurement noise power. Q(k) is an n x n matrix containing similar a priori information on the random excitation. Note that for the single input case, $E\{\underline{u}(k)\underline{u}^T(k)\}$ is a scalar which has been given the symbol $\Omega$ in the development to follow. Under assumptions of

stationarity of input excitation and measurement noise statistics, $\Omega$ is a constant and R is a constant matrix. For a scalar observation R is also a scalar. It is further assumed that excitation and measurement noise are statistically independent. A block diagram of filter operations is shown in Figure 1-2.

The derivation which leads from the definitions of P(k/k) and P(k+1/k), i.e. equations 1-6 and 1-7, to the recursive equations 1-8, 1-9, and 1-11, has been done in many ways by many authors since 1960 and will not be repeated here [4, 7]. However, it will be shown that the recursive equations to be derived in Chapter 3 which account for filter degradation, reduce to the original Kalman equations when plant and filter models coincide, and the gain matrix is computed so as to minimize estimation errors.

## THE PROBLEM STATEMENT

The problem under consideration can now be stated as; Given a plant most accurately described by equations 1-2 and 1-3, what filter performance degradation results from the implementation of the Kalman filter equation 1-4, when the gain sequence calculated using equations 1-8, 1-9 and 1-11 is based on a model of the plant which is incorrect in its representation of the plant dynamics?

CHAPTER 2

A REVIEW OF RECENT INVESTIGATIONS

The practical difficulties encountered when attempting to identify a correct (or "best") mathematical model of a plant or process to be observed are not treated here. It is assumed that the identification has been done but is subject to errors or inaccuracies. The Kalman filter performance may be degraded by errors in any of the several quantities used in the calculation of the weighting G(k). (See equations 1-8, 1-9, 1-11). Numerous investigators have considered this problem; some of their results are summarized and commented upon below. Methods for practical implementation or error analysis have been included in some cases.

In 1964 Fagin reported a generalized error analysis which included recursive equations for computing the incremental change in the covariance matrix when the filtering is done with an incorrect state transition matrix, and incorrect a priori noise statistics are used in computing gain [2]. The analysis allows a time varying observability matrix and sample interval. The assumed form of the plant in Fagin's investigation is enough different from the form assumed here that no attempt has been made to modify his results to fit the framework of the problem given in Chapter 1. A rough interpretation of those results using the notation of this paper, would be the effect of errors in $\Phi(T)$, $Q(k)$ and $R(k)$ matrices.

The recursive equations must be provided with starting values. The estimate $\hat{\underline{x}}(0/0)$ must be provided as well as P(0/0) for the first gain calculation. Whenever possible, the values used for $\hat{\underline{x}}(0/0)$ should be typical of what might be expected for the first observation. For instance if the output state of a system is thought to have zero mean, $\hat{x}_1(0/0)$ should be set to zero. The initial covariance matrix P(0/0) must reflect some level of confidence in the initial filter state. Nishimura has defined an error matrix which is the difference between the actual covariance and that calculated by the filter [8]. He has shown that if the error matrix is non-negative definite, the actual covariance of estimation error is bounded by the covariance that is calculated using the Kalman recursive equations for the optimum filter, i.e., equations 1-8, 1-9, and 1-11. This suggests that the trace of P(0/0) be given large values to ensure that the trace of the error matrix is non-negative. If application is restricted to system models which are fixed and uniformly completely observable and controllable, in the control theory sense, then according to Kalman, the calculated covariance matrix will converge to some constant matrix after enough samples [6]. The number of iterations required to reach steady state also depends on P(0/0). If this matrix is initialized with overly "pessimistic" values to satisfy Nishimura's stability condition, the filter may take too long to reach steady state in a given application. Therefore, when filter "settling time" is critical, the initialization of P(0/0)

requires some additional knowledge of the variance of the plant states so that stability may be ensured without unnecessarily increasing settling time. For this investigation the initial filter state $\underline{\hat{x}}(0/0)$ is set to the same value as the plant initial vector and P(0/0) is set to the zero matrix.

Normally stationary statistics are assumed for the input excitation and measurement noise, making $\Omega$ and R constant matrices. Errors in these quantities directly affect the elements of the steady state calculated covariance matrix. In 1966, Heffes reported on the effects of both incorrect initial covariance matrix and incorrect noise statistics in the model [3]. He includes recursive expressions for calculated covariance and gain matrices based on the false values. Results of a computer simulation of a numerical example showed the variance of the first two states of a third order system as calculated from the equations was always larger than that actually being attained, the latter of course being still greater than the optimum, given the correct model. In order to better isolate the effects of erroneous identification of plant dynamics, $\Omega$ and R will henceforth be assumed known and constant.

During this investigation, the authors became aware of very similar work being performed by S.R. Neal at NOTS China Lake. When they are published, comparison of mathematical results of the two investigations should reveal only differences in notation and assumption on the

form of the plant model. Both consider plant dynamics as being misidentified in some way, resulting in errors in the assumed state transition matrix. By isolating this type of error, it may be possible to learn which of the types of error discussed in this chapter would have the most degrading effect on filter performance in a given application. Perhaps greater emphasis could then be placed on elimination of certain types of error when establishing the system mathematical model.

The effects of any of the identification errors discussed above can be found by producing a set of recursive expressions which will produce the matrix of actual error covariance $P_a(k/k)$ and then comparing this with the covariance matrix $P_c(k/k)$ produced by using the normal Kalman equations. Another very important comparison is that between $P_a(k/k)$ and the optimum result $P_o(k/k)$ obtained when the plant model is known exactly. The difference between these last two quantities gives the true "cost" of plant mis-identification in terms of variance of estimation error, and is the measure of degradation used in this paper. These quantities are formally defined and a set of equations is developed for $P_a(k/k)$ in Chapter 3.

## CHAPTER 3

### THE PROBLEM DEVELOPMENT

As the first step toward a solution to the problem of filter performance degradation, a measure of degradation must be formally defined, along with the various covariances, according to the manner in which they were obtained. These definitions are as follows;

The measure of filter performance degradation to be used is defined as the trace of the difference matrix $\Delta P$ where

$$\Delta P = P_a(k/k) - P_o(k/k) \qquad \text{(steady state)} \quad (3\text{-}1)$$

$P_a(k/k)$ is a n x n matrix, the elements of which are the covariance values of actual estimation error produced by the filter when a given (and possibly suboptimal) gain sequence $G(i)$, $i = 0,1,2,\ldots\ldots k$ is used in the filter equation 1-4, and there has been mis-identification of plant dynamics. The recursive equations to be developed in this chapter will produce $P_a(k/k)$.

$P_o(k/k)$ is a n x n matrix of the covariance values of estimation error which results when the optimum gain sequence $G_o(i)$, $i = 0,1,2,\ldots\ldots k$, is used in the filter equation 1-4, and there has been <u>no</u> mid-identification of plant dynamics. Equation 1-11 produces $P_o(k/k)$ provided there are no identification errors as discussed above.

The third quantity to be defined is $P_c(k/k)$. This is a matrix of the covariance values of estimation error resulting when a given (and possibly suboptimal) gain sequence $G(i)$, $i = 0,1,2,\ldots\ldots k$, is used in the filter

equation 1-4, and there has been mis-identification of plant dynamics. $P_c(k/k)$ is a square matrix of the same dimensions as the order of the filter model. Equation 1-11 produces $P_c(k/k)$ provided there are identification errors as discussed above. The means of producing the three quantities defined above are diagrammed in Figure 3-1.

## THE OPTIMUM FILTER

An estimation problem that can be fitted to the Kalman filter framework is solvable by use of equations 1-4, 1-8, 1-9, and 1-11. This requires that the plant be perfectly described by equations 1-2 and 1-3. However, when the filter employs an incorrect model, the recursively calculated covariance (equation 1-11) is no longer optimum, so the performance index as given by equation 1-5 no longer applies. The objective of minimizing the mean squared error is still valid but the mean square error now becomes the trace of the actual covariance matrix of estimation error, $P_a(k/k)$. To distinguish the types of covariance mentioned thus far, subscripts have become necessary. In the derivation of $P_a(k/k)$ which follows, the subscript f denotes filter quantities while the subscript p refers to the most accurate mathematical model of the plant or system.

## THE SUBOPTIMUM FILTER

Suppose the matrix difference equation giving the response of a discrete system at sampling instants has been identified as

$$\underline{x}(k+1) = \Phi_f(T)\underline{x}(k)+\Gamma_f(T)\underline{u}(k) \tag{3-2}$$

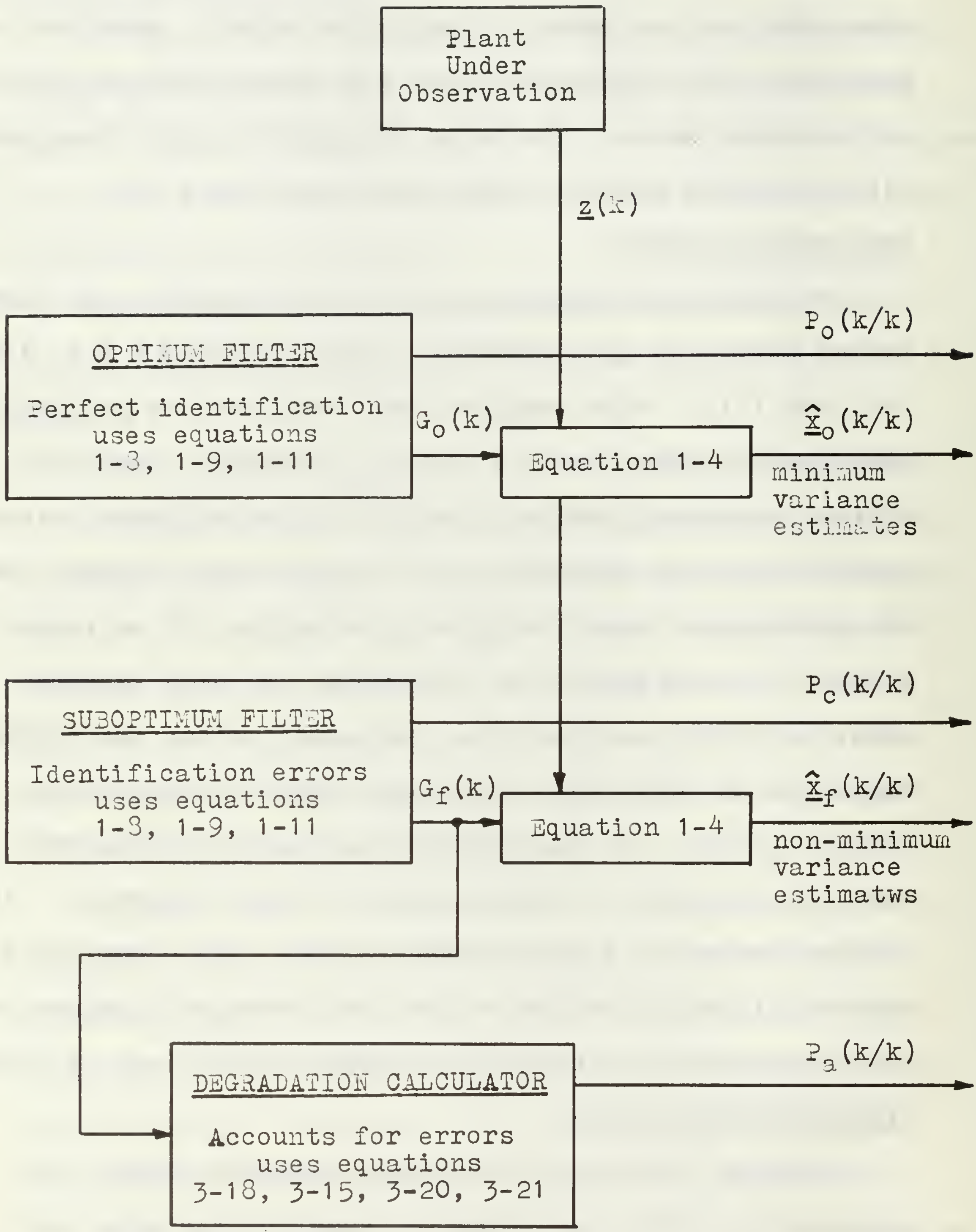


Fig. 3-1 The Three Types of Calculated Covariance

when the most accurate description of the same system is given by,

$$\underline{x}(k+1) = \Phi_p(T)\underline{x}(k)+\Gamma_p(T)\underline{u}(k) \quad (3\text{-}3)$$

where

$$\Phi_f(T)+\delta\Phi(T) = \Phi_p(T);\ \Gamma_f(T)+\delta\Gamma(T) = \Gamma_p(T) \quad (3\text{-}4)$$

Assume the observation in either case is given by

$$z(k) = H\underline{x}(k)+\underline{v}(k) \quad (3\text{-}5)$$

If the Kalman filter equations 1-8 through 1-10 were to be used, the filtering would be suboptimal, i.e., equation 1-5 would not be minimized. This is readily seen by noting that equation 1-8 becomes

$$P(k+1/k) = \Phi_f(T)P(k/k)\Phi_f^T(T)+Q_f \quad (3\text{-}6)$$

and being independent of observed data, cannot reflect errors in $\Phi(T)$. The calculated gain which depends upon equation 3-6 would therefore be suboptimum.

THE ACTUAL COVARIANCE MATRIX

The errors $\delta\Phi(T)$ and $\delta\Gamma(T)$ are taken into account as follows:

Assume a plant has been misidentified as in the last section, equation 3-2, and a Kalman filter applied. The filter equation would be;

$$\underline{\hat{x}}(k+1/k+1) = \Phi_f(T)\underline{\hat{x}}(k/k)+G(k+1)[\underline{Z}(k+1)-H\ \Phi_f(T)\underline{\hat{x}}(k/k)] \quad (3\text{-}7)$$

If this equation and the correct plant description, equation 3-3, are substituted into the appropriate quantities of the definition for $P(k/k)$, equation 1-6, the resulting expression becomes $P_a(k/k)$. This can be shown as follows (reducing the index by one and dropping T from $\Phi(T)$ and $\Gamma(T)$ ),

$$\underline{x}(k)-\hat{\underline{x}}(k/k) = \Phi_p\underline{x}(k-1)+\Gamma_p\underline{u}(k-1)-\Phi_f\hat{\underline{x}}(k-k/k-1)$$

$$-G(k)[\underline{z}(k)-H\Phi_f\hat{\underline{x}}(k/k)] \tag{3-8}$$

but $\Phi_p = \Phi_f+\delta\Phi$

and for convenience define

$$\Phi_f-G(k)H\Phi_f\equiv\Phi_f^* \tag{3-9}$$

$$\delta\Phi-G(k)H\delta\Phi\equiv\delta\Phi^* \tag{3-10}$$

$$\Gamma_p-G(k)H\Gamma_p\equiv\Gamma_p^* \tag{3-11}$$

then after some manipulation, equation 1-6 becomes,

$$\begin{aligned}
P_a(k/k) = {} & \Phi_f^*P_a(k-1/k-1)\Phi_f^{*T} \\
& +\Phi_f^*E\{\underline{x}(k-1)\underline{x}^T(k-1)\}\delta\Phi^{*T} \\
& -\Phi_f^*E\{\hat{\underline{x}}(k-1/k-1)\underline{x}^T(k-1)\}\delta\Phi^{*T} \\
& +\delta\Phi^*E\{\underline{x}(k-1)\underline{x}^T(k-1)\}\Phi_f^{*T} \\
& -\delta\Phi^*E\{\underline{x}(k-1)\hat{\underline{x}}^T(k-1/k-1)\}\Phi_f^{*T} \\
& +\delta\Phi^*E\{\underline{x}(k-1)\underline{x}^T(k-1)\}\delta\Phi^{*T} \\
& +\Gamma_p^*E\{\underline{u}(k-1)\underline{u}^T(k-1)\}\Gamma_p^{*T} \\
& +G(k)E\{\underline{v}(k)\underline{v}^T(k)\}G^T(k)
\end{aligned} \tag{3-12}$$

Now, taking the definition, equation 1-7, reducing the index by one and noting that

$$\hat{\underline{x}}(k/k-1) = \Phi_f\hat{\underline{x}}(k-1/k-1) \tag{3-13}$$

then upon substitution of appropriate quantities, equation

1-7 becomes

$$\begin{aligned} P_a(k/k-1) = {} & \Phi_f P_a(k-1/k-1)\Phi_f^T \\ & +\Phi_f E\{\underline{x}(k-1)\underline{x}^T(k-1)\}\delta\Phi^T \\ & -\Phi_f E\{\underline{\hat{x}}(k-1/k-1)\underline{x}^T(k-1)\}\delta\Phi^T \\ & +\delta\Phi E\{\underline{x}(k-1)\underline{x}^T(k-1)\}\Phi_f^T \\ & -\delta\Phi E\{\underline{x}(k-1)\underline{\hat{x}}^T(k-1/k-1)\}\Phi_f^T \\ & +\delta\Phi E\{\underline{x}(k-1)\underline{x}^T(k-1)\}\delta\Phi^T \\ & +\Gamma_p E\{\underline{u}(k-1)\underline{u}^T(k-1)\}\Gamma_p^T \end{aligned} \tag{3-14}$$

Comparison of equation 3-12 with equation 3-14 and the use of the definitions in equations 3-9, 3-10, and 3-11, reveals that

$$\begin{aligned} P_a(k/k) = {} & P_a(k/k-1)-G(k)HP_a(k/k-1)-P_a(k/k-1)H^T G^T(k) \\ & +G(k)[HP_a(k/k-1)H^T+R]G^T(k) \end{aligned} \tag{3-15}$$

Kalman has shown that if gain is calculated from equation 1-9, the trace of the right hand side of equation 3-15 is minimized, and the recursive equation 1-11 is obtained. It can be concluded here that given a known error $\delta\Phi$, the recursive equations which would be used for minimum variance estimates would be 1-9, 3-7, 1-11 and 3-14.

That is to say, if a Kalman filter is to be applied with a known error $\delta\Phi$ in the model of plant dynamics, then minimum variance estimates can still be produced, provided the error $\delta\Phi$ is taken into account by use of equations 1-9, 3-7, 1-11 and 3-14. Except for the case of intentional

mis-modeling for the sake of order reduction, the error $\delta\Phi$ would of course be used to correct $\Phi_f$ and the original optimal Kalman filter equations 1-8, 1-9 and 1-11 would be used.

THE RECURSIVE CALCULATION OF ACTUAL COVARIANCE

Equation 3-15 is entirely suitable for use as a recursive expression for computer simulation. However, equation 3-14 must be adapted from its present form to one which avoids explicit use of the expectation operation. The approach taken was to define the matrix quantities

$$D(k) \equiv E\{\underline{x}(k)\underline{x}^T(k)\} \tag{3-16}$$

$$K(k) \equiv E\{\hat{\underline{x}}(k/k)\underline{x}^T(k)\} \tag{3-17}$$

Matrix algebra and the advance of index yields (from equation 3-14)

$$\begin{aligned} P(k+1/k) = {} & \Phi_f P(k/k)\Phi_f^T + \Phi_f D(k)\delta\Phi^T - \Phi_f K(k)\delta\Phi^T \\ & + \delta\Phi^T D^T(k)\Phi_f^T - \delta\Phi K^T(k)\Phi_f^T + \delta\Phi D(k)\delta\Phi^T + Q_p \end{aligned} \tag{3-18}$$

Equation 3-18 is in usable form, but requires recursive expressions for D(k) and K(k). These are obtained from the definitions (equations 3-16 and 3-17):

$$D(k+1) \equiv E\{\underline{x}(k+1)\underline{x}^T(k+1)\} = E\{[\Phi_p\underline{x}(k)+\Gamma_p\underline{u}(k)][\Phi_p\underline{x}(k)+\Gamma_p\underline{u}(k)]^T\} \tag{3-19}$$

Expanding the right hand side and noting that $\underline{u}(k)$ and $\underline{x}(k)$ are uncorrelated,

$$\begin{aligned} D(k+1) &= \Phi_p E\{\underline{x}(k)\underline{x}^T(k)\}\Phi_p^T + \Gamma_p E\{\underline{u}(k)\underline{u}^T(k)\}\Gamma_p^T \\ &= \Phi_p D(k)\Phi_p^T + Q_p \end{aligned} \tag{3-20}$$

Similar manipulations with the definition of K(k+1) yield

$$K(k+1) \equiv E\{\underline{\hat{x}}(k+1/k+1)\underline{x}^T(k+1)\}$$

$$=[I-G(k+1)H]\Phi_f K(k)\Phi_p^T+G(k+1)HD(k+1) \qquad (3\text{-}21)$$

The iterative expressions derived are now summarized in the proper order for calculation:

$$P(k+1/k) = \Phi_f P(k/k)\Phi_f^T+\Phi_f D(k)\delta\Phi^T-\Phi_f K(k)\delta\Phi^T$$
$$+\delta\Phi D(k)\Phi_f^T-\delta\Phi K^T(k)\Phi_f^T+\delta\Phi D(k)\delta\Phi^T$$
$$+Q_p \qquad (3\text{-}18)$$

$$G(k+1) = P(k+1/k)H^T[HP(k+1/k)H^T+R]^{-1} \qquad (1\text{-}9)$$

$$P(k+1/k+1) = P(k+1/k)-G(k+1)HP(k+1/k)-P(k+1/k)H^TG^T(k+1)$$
$$+G(k+1)[HP(k+1/k)H^T+R]G^T(k+1) \qquad (3\text{-}15)$$

$$D(k+1) = \Phi_p D(k)\Phi_p^T+Q_p \qquad (3\text{-}20)$$

$$K(k+1) = [I-G(k+1)H]\Phi_f K(k)\Phi_p^T+G(k+1)HD(k+1) \qquad (3\text{-}21)$$

Several comments on the appearance of equation 1-9 in this list are appropriate at this time. First, if the plant is correctly identified i.e., $\Phi_f = \Phi_p$, $\delta\Phi = 0$, then it is obvious that equation 3-18 reverts to 1-8 and equation 3-15 is of course equation 1-10. Equations 3-20 and 3-21 would still exist but would not be used in 3-18, therefore standard Kalman filtering results. Second, if the plant is mis-identified, the use of equation 1-9 in the order shown will produce the set of minimum variance estimates to be used in the case of order reduction mis-modeling, mentioned above. Third, if any other gain sequence is produced externally to equations 3-18, 3-15, 3-20 and 3-21 then

equation 3-15 will give the actual covariance of estimation error that would result when the gain sequence supplied is utilized in the Kalman filter equation 1-4.

THE SENSITIVITY TO ERRORS IN PLANT DYNAMICS

One of the original objectives of this investigation was to find an analytic expression for the sensitivity of the performance index J to plant identification errors. This would be of the form,

$$dJ = \frac{\partial J}{\partial \alpha_1}\, d\alpha_1 + \frac{\partial J}{\partial \alpha_2}\, d\alpha_2 + \cdots$$

where $\alpha_i$ is one of the plant parameters subject to misidentification. The development of such an expression would involve finding total differentials for each of the trace elements of $P_o$ (steady state) and then adding to get dJ. If the filter is stable and $P_o(k/k)$ eventually reaches a constant value, an implicit expression for the steady state covariance is easy to obtain by setting $P_o(k+1/k+1)$ equal to $P_o(k/k)$. The difficulty lies in the amount of algebra involved when the system order is two or greater. Each partial derivative of an element of the covariance matrix is a function of all the other elements.

To then find partial derivatives of the steady state covariance matrix trace elements, P (k/k) is considered along with its definition,

$$\begin{aligned} P(k/k) &\equiv E\{\underline{x}(k)\underline{x}^T(k) - \underline{\hat{x}}(k/k)\underline{x}^T(k) - \underline{x}(k)\underline{\hat{x}}^T(k/k) + \underline{\hat{x}}(k/k)\underline{\hat{x}}^T(k/k)\} \\ &\equiv D(k) - K(k) - K^T(k) + E\{\underline{\hat{x}}(k/k)\underline{\hat{x}}^T(k/k)\} \end{aligned} \qquad (3\text{-}22)$$

The sum of the terms on the right hand side reaches a constant or steady state value. Moreover, it can be shown that in a stable system in which $\Phi(T) \neq I$ each of the terms in the sum becomes constant. For example, in the stable time invariant plant with feedback, the average "power" in the states D(k) becomes a constant times the driving "power".

Therefore, D(k+1) is set equal to D(k); an implicit function is obtained and $\frac{\partial D_{ij}}{\partial \alpha_n}$, where $\alpha_n$ is a plant parameter, can be found. The partial derivative $\frac{\partial P_{ij}}{\partial \alpha_n}$ will be the sum of the similar quantities on the right hand side of equation 3-22. The procedure is straightforward, but the amount of algebra is prohibitive. No better method was found.

EXAMPLES

The examples which follow will serve to demonstrate how rapidly algebraic complication can arise with slight increases in plant complexity. All are scalar cases, making the performance index J equal to the steady state covariance P. For the two simplest examples a sensitivity function is calculated, as well as the actual filter degradation expression. For the low pass filter example, a means of obtaining the sensitivity function is discussed, but it is not done. In that example only the degradation expression is included as a function of the plant parameter.

Example 1: A Simple Amplifier

Consider the plant shown in figure 3-2. The state x at the kth sampling instant is given as x (k) = a u (k). By comparison with the usual state space discrete notation

it can be seen that $\Phi(T) = 0$; $\Gamma(T) = a$

The kth observation is written as $Z(k) = x(k)+v(k)$

Suppose that optimal estimates $\hat{x}(k/k)$ of the state x are required. Then application of equation (1-4) yields

$$\hat{x}(k/k) = G(k)Z(k) \tag{3-23}$$

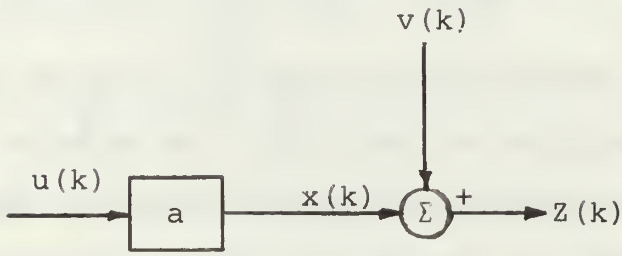


Fig. 3-2

The Simple Amplifier

Applying equations (1-8), (1-4), and (1-11) for optimum filtering, the recursive sequence becomes

$$\text{Filter gain: } G(k+1) = \frac{a^2\Omega}{a^2\Omega+R} \tag{3-24}$$

$$\text{Error Covariance (variance)}: P(k/k) = \frac{Ra^2\Omega}{a^2\Omega+R} \tag{3-25}$$

$$\text{Conditional Error variance: } P(k+1/k) = Q=a^2\Omega \tag{3-26}$$

Substitution of the expression for gain into equation 3-23 yields

$$\hat{x}(k/k) = \frac{a^2\Omega}{a^2\Omega+R} Z(k) \tag{3-27}$$

Recalling that $\Omega$ is the variance of the perturbation and R is the variance of the measurement noise, the quantity $a^2\Omega$ could be thought of as the average signal "power" and R as the average noise "power", making the optimal weighting

$$\frac{\text{signal power}}{\text{signal power + noise power}}$$

which satisfies intuition for the case of observing a signal in noise.

The sensitivity function for this example can be found easily by differentiating equation 3-25 with respect to the plant parameter.

$$\frac{dP}{da} = \frac{2R^2\Omega a}{(a^2\Omega+R)^2} \qquad (3\text{-}28)$$

Now suppose that the true plant is as shown in figure 3-2, but that the amplification has been incorrectly identified as $a_f$ where $a=a_f + \delta a$. Application of the Kalman filter equations then gives a calculated gain

$$G_c = \frac{a_f^2\Omega}{a_f^2\Omega+R} \qquad (3\text{-}29)$$

If this gain is used to estimate x, the degradation due to misidentification can be found as the difference between the covariance resulting from using equations 3-18 and 3-15 and the optimum value. Substitution of $G_c$ into equations 3-18 and 3-15 yields

$$P_a(k/k) = \frac{R(a_f^4\Omega+Ra^2)}{(a_f^2\Omega+R)^2} \qquad (3\text{-}30)$$

Note than when $a_f$= a this reduces to equation 3-25, as required. The degradation in performance P is therefore

obtained by subtracting the right hand side of equation 3-25 from $P_a(k/k)$ as given in equation 3-30.

$$\text{Degradation } \Delta P = P_a - P_o$$

$$\Delta P = \frac{R^2\Omega^2(a^2 - a_f^2)^2}{(a_f^2\Omega + R)^2(a^2\Omega + R)} \tag{3-31}$$

Figure 3-3 shows degradation in the performance index J as a per cent of the optimum versus percentage error in the plant parameter. The values used were

$$R = .01;\ \Omega = 1.0;\ a_f = 20.0$$

Degradation in this example is very slight; an error of 30% in identification degrades the filter performance by only .00133%. This may be partially explained by noting that the degradation function shown is approximately directly proportional to the square of the measurement noise variance, and that a very low value was chosen for R. Nevertheless, it can be concluded that the Kalman filter performance is not very sensitive to plant identification in this application.

Example 2. Integrator-Amplifier

As another example of only slightly greater difficulty, consider the plant described by the transfer function

$$\frac{x(s)}{u(s)} = \frac{a}{s} \tag{3-32}$$

Figure 3-4 shows the discrete representation of this plant.

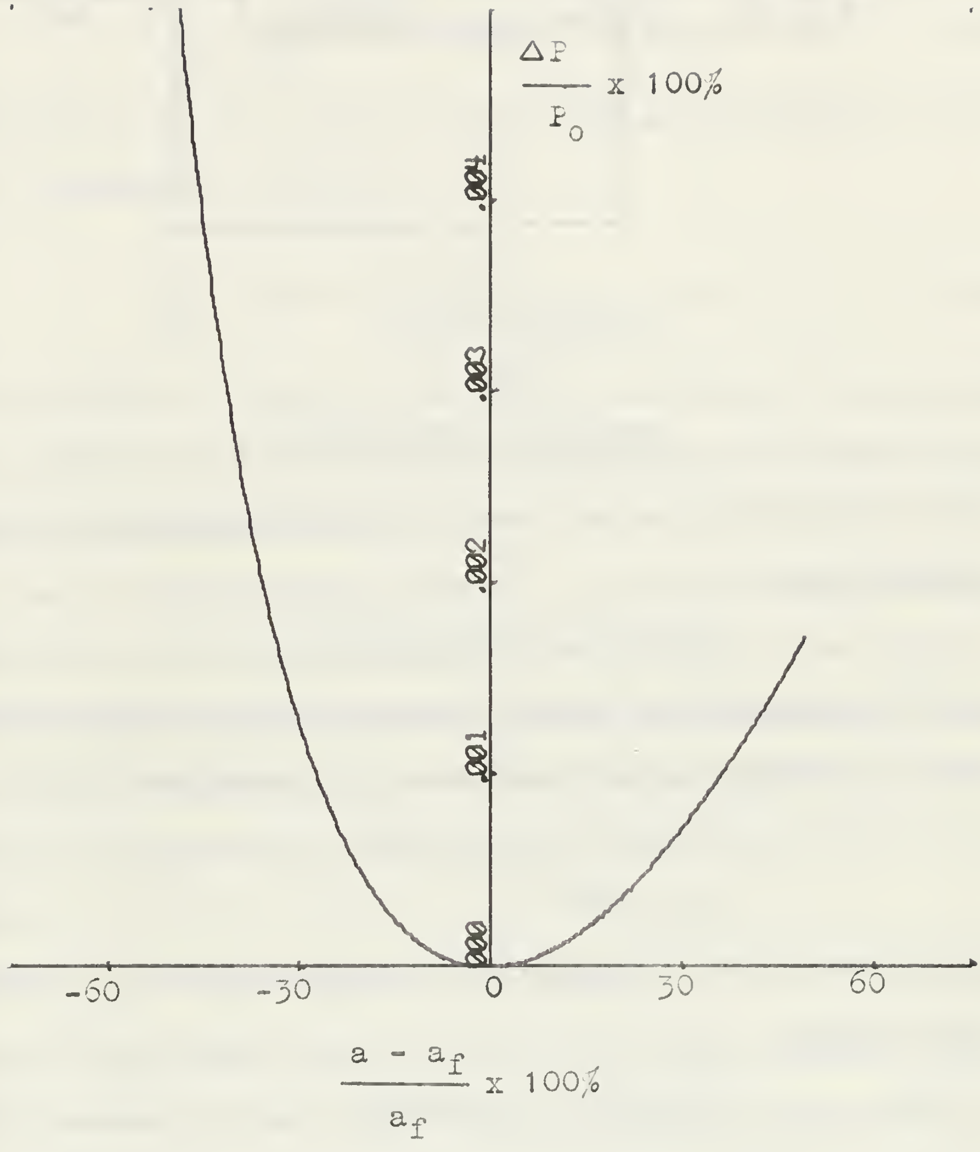


Fig. 3-3 Degradation for the Simple Amplifier

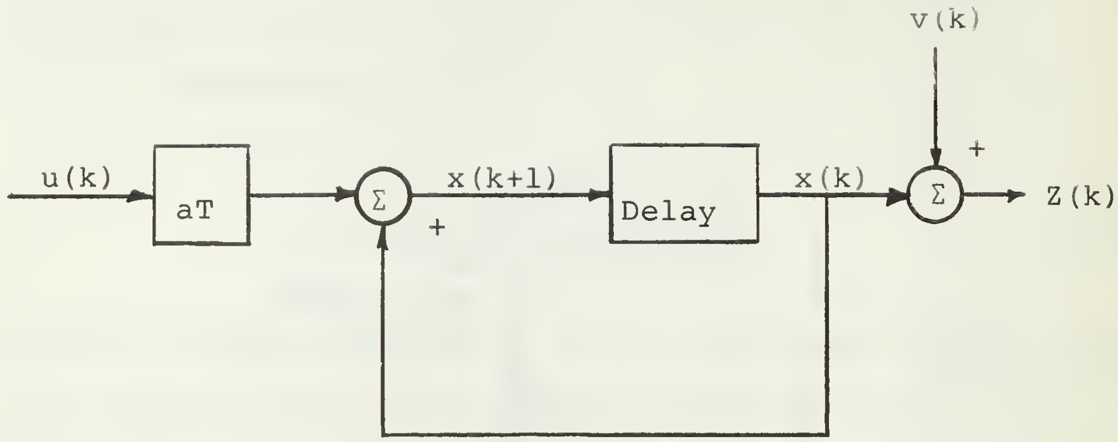


Fig. 3-4

Integrator-Amplifier

The difference equation describing the response at sampling instants is

$$x(k+1) = x(k) + aTu(k) \tag{3-33}$$

with the observation again consisting of a single state plus noise. Examination of equation 3-33 reveals that

$$\Phi(T) = 1; \; \Gamma(T) = aT$$

$$Q = \Omega a^2 T^2 \tag{3-34}$$

The Kalman filter equations 1-8, 1-9 and 1-11 become, respectively

$$P(k+1/k) = P(k/k) + Q \tag{3-35}$$

$$G(k+1) = \frac{P(k+1/k)}{P(k+1/k)+R} \tag{3-36}$$

$$P(k+1/k+1) = \frac{RP(k+1/k)}{P(k+1/k)+R} \tag{3-37}$$

The steady state covariance can be found by equating P (k+1/k+1) to P(k/k). As in the previous example, this is optimum filtering when identification of the plant

parameter a is perfect. The resulting covariance is the solution to a quadratic equation, viz.,

$$P_o = \frac{1}{2}\left[\sqrt{Q^2+4RQ}-Q\right] \tag{3-38}$$

Substitution for Q yields

$$P_o = \frac{1}{2}\left[\sqrt{\Omega^2 a^4 T^4+4R\Omega a^2 T^2}-\Omega a^2 T^2\right] \tag{3-39}$$

Sensitivity of the steady state optimum covariance to the plant parameter a becomes

$$\frac{dP}{da} = a\Omega T^2 \left[\frac{\Omega a^2 T^2+2R}{\sqrt{\Omega^2 a^4 T^4+4R\Omega a^2 T^2}} -1\right] \tag{3-40}$$

The sensitivity function was easily obtained in this example, and could be used to determine degradation for small perturbations in the parameter a.

As in the previous example, it is now assumed that $a_f$ was used as the amplification value in the filter model, and the gain sequence resulting from the misidentification is known. The final value of the filter gain could be found by manipulation of equations 3-35, 3-36 and 3-37 to yield

$$G_f = \frac{1}{2R}\left[\sqrt{Q_f^2+4RQ_f}-Q_f\right] \tag{3-41}$$

This is the steady state value of the gain sequence used for the erroneous filter models and therefore can be used to find the actual value of steady state covariance. The actual steady state covariance is again found by proper substitutions in equations 3-18 and 3-15 to be

$$P_a + \frac{(1-G_f)^2 Q_p+G_f^2 R}{2G_f-G_f^2} \tag{3-42}$$

Comparison of equations 3-41 and 3-38 reveals that if identification were perfect, the optimum steady state covariance would be

$$P_o = G_o R \tag{3-43}$$

The degradation due to identification error then becomes

$$\Delta P = P_a - P_o$$

$$\Delta P = \frac{(1-G_f)^2 Q_p + G_f^2 R(1+G_o) - 2G_o G_f R}{2G_f - G_f^2} \tag{3-44}$$

From equation 3-34

$$Q_p = \Omega a^2 T^2 \tag{3-45}$$

If equation 3-45 is substituted for $Q_f$ in equation 3-41 the final value of the optimum gain sequence results

$$G_o = \frac{1}{2R}[\sqrt{\Omega^2 a^4 T^4 + 4R\Omega a^2 T^2} - \Omega a^2 T^2] \tag{3-46}$$

Again using equation 3-34 to obtain $Q_f$ and substitution the result into equation 3-41 yields

$$G_f = \frac{1}{2R}[\sqrt{\Omega^2 a_f^4 T^4 + 4R\Omega a_f^2 T^2} - \Omega a_f^2 T^2] \tag{3-47}$$

Substitution of equations 3-45, 3-46, and 3-47 into equation 3-44 gives the degradation as a function of a and $a_f$. It can be shown that equation 3-44 becomes zero as required, when $G_o = G_f$. A graph of equation 3-44 is shown in figure 3-5. The constants used were

$$R = .01;\ \Omega = 1.0;\ a_f = 20.0$$

As in the previous example, degradation is not very great with considerable errors in identification, for the a priori noise statistics chosen.

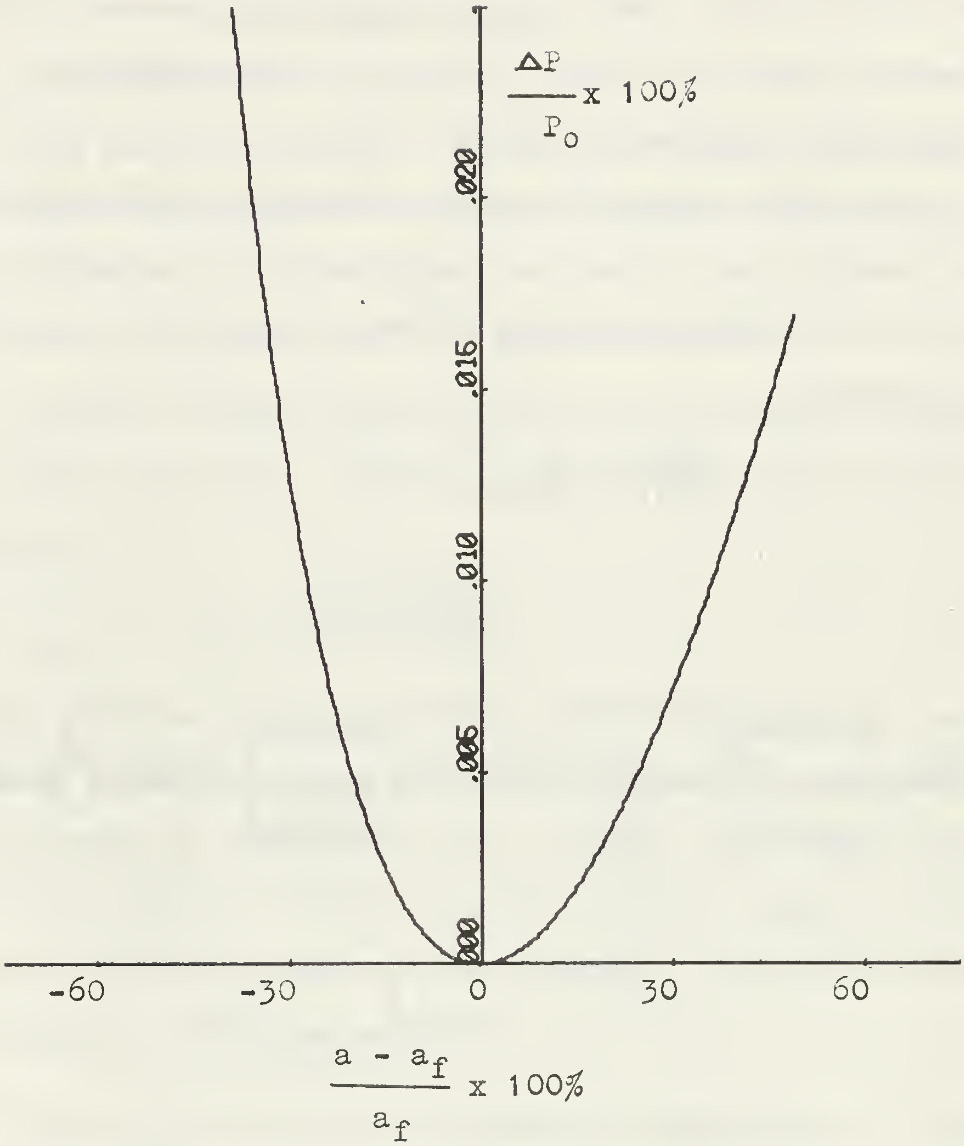


Fig. 3-5 Degradation for the Integrator-Amplifier

A final comment on this example is that although the assumption that the individual terms on the right hand side of equation 3-22 become constant does not hold for this plant, the results obtained by using equations 3-15 and 3-18 are still valid. This will be the case whenever $\Phi(T)$ is the identity matrix or unity as it is in this example.

Example 3. Low Pass Filter

The plant shown in Figure 3-6 represents a more meaningful example and is not too complicated for algebraic analysis. This is the discrete model for the continuous system transfer function.

$$\frac{x(s)}{u(s)} = \frac{a}{s+a}$$

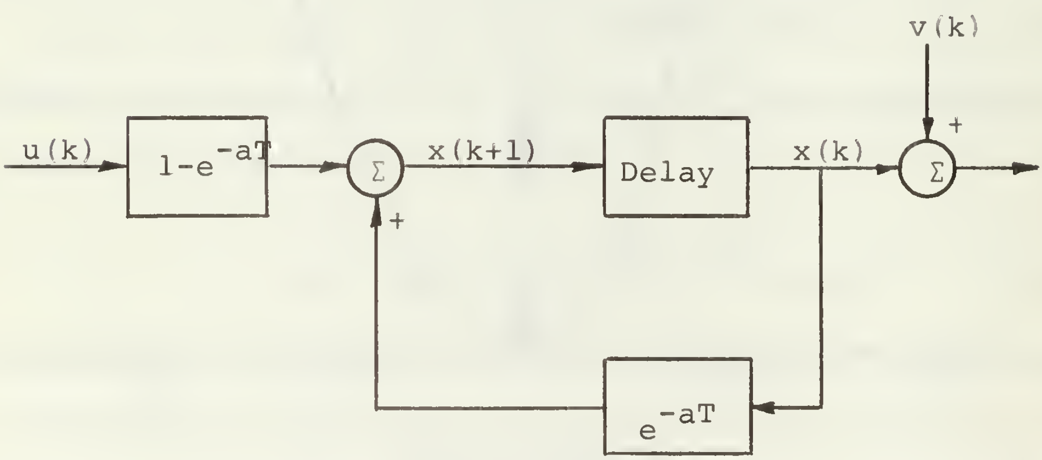


Fig. 3-6

Low Pass Filter

The difference equation describing the plant is

$$x(k+1) = e^{-aT}x(k)+(1-e^{-aT})u(k) \tag{3-48}$$

therefore $\Phi(T) = e^{-aT}$; $\Gamma(T) = 1-e^{-aT}$

The observation is again

$$z(k) = x(k) + v(k)$$

The plant parameter is again a.

As in the previous example, it will be assumed that mis-identification has resulted in an erroneous filter model, i.e.,

$$\Phi_f(T) = e^{-a_f T}; \; \Gamma_f(T) = 1-e^{-a_f T}$$

It is further assumed that the gain calculation is based on the Kalman equations, resulting in the following steady state gain:

$$G_f = \frac{\Phi_f{}^2 P_c+Q_f}{\Phi_f{}^2 P_c+Q_f+R} \tag{3-49}$$

Now the effects of the erroneous identification can be found by using equations 3-18, 3-15, 3-20 and 3-21, along with the gain as given in equation 3-49. However the steady state value of P(k/k) is required which for this example is the solution of a quadratic scalar equation. From the Kalman equations it can be shown that

$$P_c = \frac{1}{2\Phi_f{}^2}[R\Phi_f{}^2-Q_f-R+\sqrt{(Q_f+R-R\Phi_f{}^2)^2+4RQ_f\Phi_f{}^2}] \tag{3-50}$$

Substitution of equation 3-50 into 3-49 yields an expression for $G_f$ in terms of plant variables only.

$$G_f = \frac{R\Phi_f{}^2+Q_f-R+\sqrt{(\Phi_f+R-R\Phi_f{}^2)^2+4RQ_f\Phi_f{}^2}}{R\Phi_f{}^2+Q_f+R+\sqrt{(Q_f+R-R\Phi_f{}^2)^2+4RQ_f\Phi_f{}^2}} \tag{3-51}$$

It should be noted that a sensitivity function $\frac{dP}{da}$ could have been obtained by solving for P(k/k) in equation 3-49 substituting for the steady state gain from equation 3-51, and then forming $\frac{\partial P_c}{\partial Q_f}$, $\frac{\partial P}{\partial \Phi_f}$, $\frac{\partial Q_f}{\partial a}$ and $\frac{\partial \Phi_f}{\partial a}$. However, the expressions obtained are unwieldy and reveal little insight into the problem of filter degradation. The sensitivity function approach has the further limitation of small parameter variations whereas the application of equations 3-18, 3-15, 3-20, 3-21 does not. If the gain as given by equation 3-51 is used for the filter, equations 3-18, 3-15, 3-20, 3-21 give the conditional covariance of estimation error as the following

$$P_a(k+1/k) = \Phi_f^{\,2} P_a(k/k) + (\Phi_p^2 - \Phi_f^2)\, D(k) - 2\Phi_f(\Phi_p - \Phi_f)K(k) + Q_p \tag{3-52}$$

Again, it is obvious that when plant model and filter model coincide, the result is the Kalman equation for conditional covariance. Proceeding to the expression for the steady state covariance of estimation error, one obtains

$$P_a = \frac{1-2G_f+G_f^{\,2}}{G_f(2-G_f)}[(\Phi_p^{\,2}-\Phi_f^{\,2})D_a - 2\Phi_f(\Phi_p-\Phi_f)K_a + Q_p] \tag{3-53}$$

Where $D_a$ and $K_a$ are the steady state values of $E\{x^2\}$ and $E\{x\hat{x}\}$ respectively. These are found by equating the values at the $(k+1)^{th}$ iteration to those for the $k^{th}$ iteration as follows:

$$D(k+1) = \Phi_p^{\,2} D(k) + Q_p; \quad D_a = \frac{Q_p}{1-\Phi_p^{\,2}} \tag{3-54}$$

$$K(k+1) = (1-G_f)\Phi_f\Phi_p K(k)+G_f D(k+1); \quad K_a = \frac{G_f Q_p}{(1-\Phi_p{}^2)(1-\Phi_f\Phi_p+G_f\Phi_f\Phi_p)} \tag{3-55}$$

Substituting equations 3-54 and 3-55 into 3-53 one obtains an expression for the actual steady state covariance in terms of the gain

$$P_a = \frac{(1-G_f)^2}{G_f(2-G_f)}\left[\frac{(\Phi_p{}^2-\Phi_f{}^2)(1-G_f\Phi_f\Phi_p)-2\Phi_f(\Phi_p-\Phi_f)G_f}{1-G_f\Phi_p\Phi_f}+Q_p\right] \tag{3-56}$$

Equation 3-56 shows that when $\Phi_p = \Phi_f$ the expression for optimum covariance would be

$$P_o = \frac{(1-G_o)^2}{G_o(2-G_o)} \tag{3-57}$$

Where $G_o$ is the steady state gain obtained by using Kalman equations with the correct model, as in equation 3-51. Making substitutions for $G_f$ and $G_o$ the degradation in filter performance can be found as

$$\Delta P = P_a - P_o$$

i.e., equation 3-56 minus equation 3-57.

CHAPTER 4

## COMPUTER SIMULATIONS

The great increase in complexity of sensitivity functions which accompanies the slightest increase in system complexity was readily evident in Chapter 3. Even a scalar case such as the low pass filter with a single pole produces unwieldy algebraic expressions for sensitivity. For systems of second order or higher, it appears to be more advantageous to perform a computer simulation of some specific case. This portion of the investigation was performed on the CDC 1604 Digital Computer and consisted of four parts. The first was a verification of the algorithm derived in Chapter 3 (equations 3-18, 3-15, 3-20 and 3-21). This algorithm, while ostensibly accurate, provides numerous opportunities for error in its implementation. The remaining simulations were investigations of specific examples to test the utility of the recursive solution in actual problems. The desired end result was a means of knowing the degradation of filter performance as a function of error in one or more plant parameters, given the filter operating parameter values. Such information, along with the knowledge (or an estimate) of the accuracy of the filter model parameters, could be useful in deciding whether more (or less) accurate identification is called for. For example, assume the model for a second order system uses a damping factor $\zeta_f$ and a natural frequency

$\omega_f$ which through some previous error analysis are known to be accurate within 10 per cent. A look at the steady state solution obtained from the recursive equations based on ten per cent errors will provide the actual degradation in filter performance if the plant parameters lie on the tolerance limit.

For this example, suppose that this amount of degradation from optimum is incompatable with the estimation accuracy requirements. By examining the results for various lower parameter errors it will become apparent to what accuracy the parameters must be identified. A flow chart for this type of investigation is shown in Figure 4-1. Given a gain sequence, an estimate of parameter error, the filter model and the correct $\Omega$, R, and H matrices; the quantities $\phi_f$, $\phi_p$, $Q_f$, and $Q_p$ can be found and used to implement a recursive sequence of equations 3-18, 3-15, 3-20 and 3-21. In all examples which follow, the filter model employs correct initialization and accurate $\Omega$ and R matrices. All are single input systems with only one observed state making $\Omega$ and R scalars, with values taken as 1.0 and 0.01 respectively.

a. VERIFICATION OF RECURSIVE SOLUTION

Equations 3-18, 3-15, 3-20 and 3-21 of Chapter 3 were verified by comparing the actual steady state covariance matrix trace with that obtained by driving a simulated plant, observing the entire state vector and computing the quantity $[(\underline{x} - \hat{\underline{x}})(\underline{x} - \hat{\underline{x}})^T]$. This was

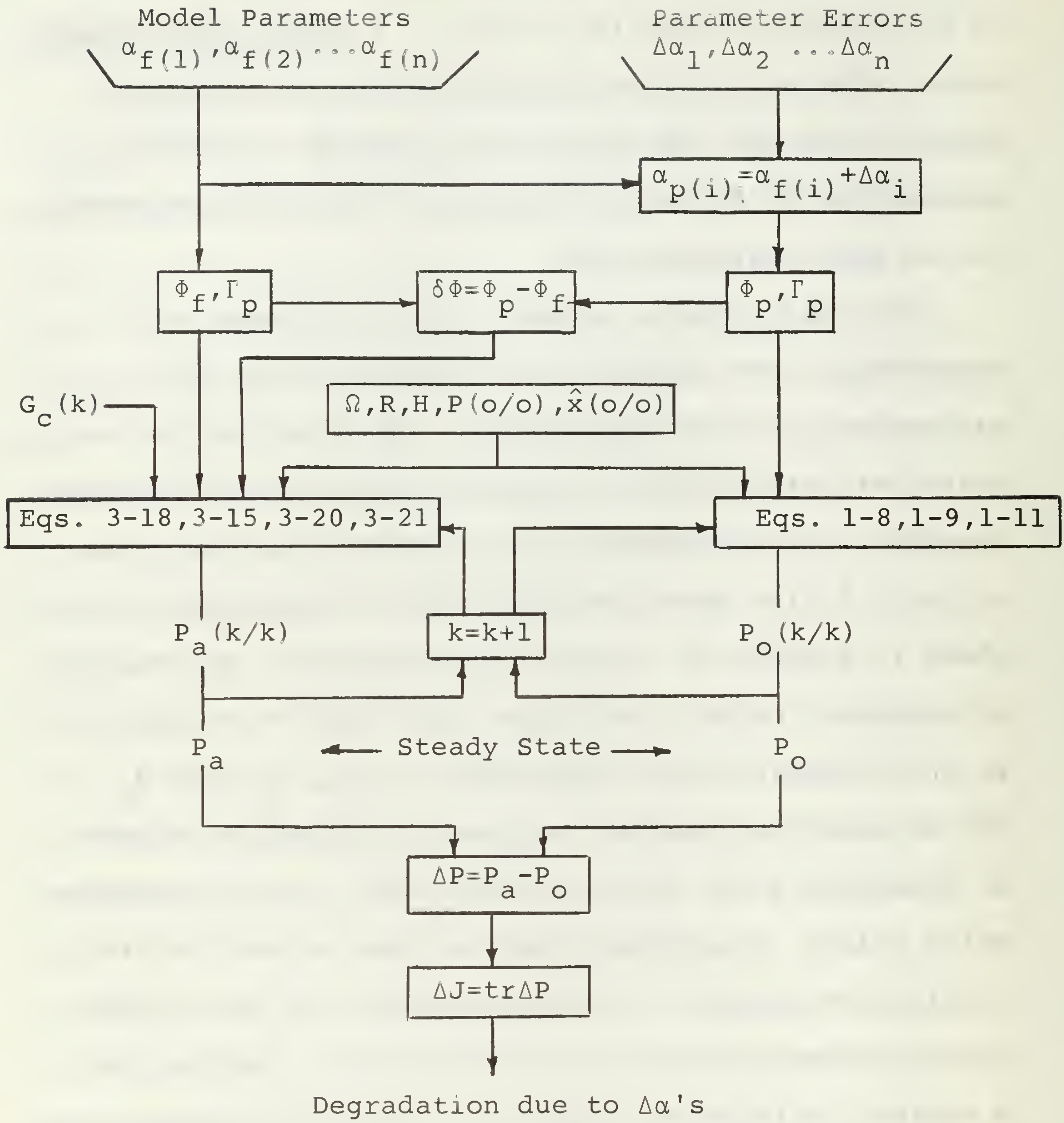


Fig. 4-1 Calculation of Filter Performance Degradation

done for each filter-plant combination for 1000 different random sequences of driving and measurement noise, preserving the average values at each successive iteration. At least 30 iterations at steady state were used. The ensemble averages were then averaged in time, giving 30 samples from which hypothesis testing could be done. The entire procedure above was repeated for numerous points including from zero to 20 per cent errors in each plant parameter in order to verify that no programming errors existed in the calculation of actual steady state covariance.

b. EXAMPLE OF TWO-PARAMETER SENSITIVITY

Next, a numerical example of two-parameter sensitivity was performed using the method outlined in Figure 4-1. The second order model was chosen to be of the form

$$\frac{\omega_f^2}{s^2 + 2\zeta_f\omega_f s + \omega_f^2} \tag{4-1}$$

with filter parameters $\zeta_f$ and $\omega_f$ taken as Cos($\pi$/4) and 10.0, respectively. The plant was assumed to be driven and sampled at 0.1 second intervals, with initial conditions zero, and with $x_1$ the only observable state. The plant parameters $\zeta_p$ and $\omega_p$ were varied from zero to $\pm$50% of those used by the filter model. For each set of plant parameters, the difference between the actual covariance trace and that which could be obtained if the filter matched the plant is computed and stored. The resulting values are points on a bowl-shaped surface which

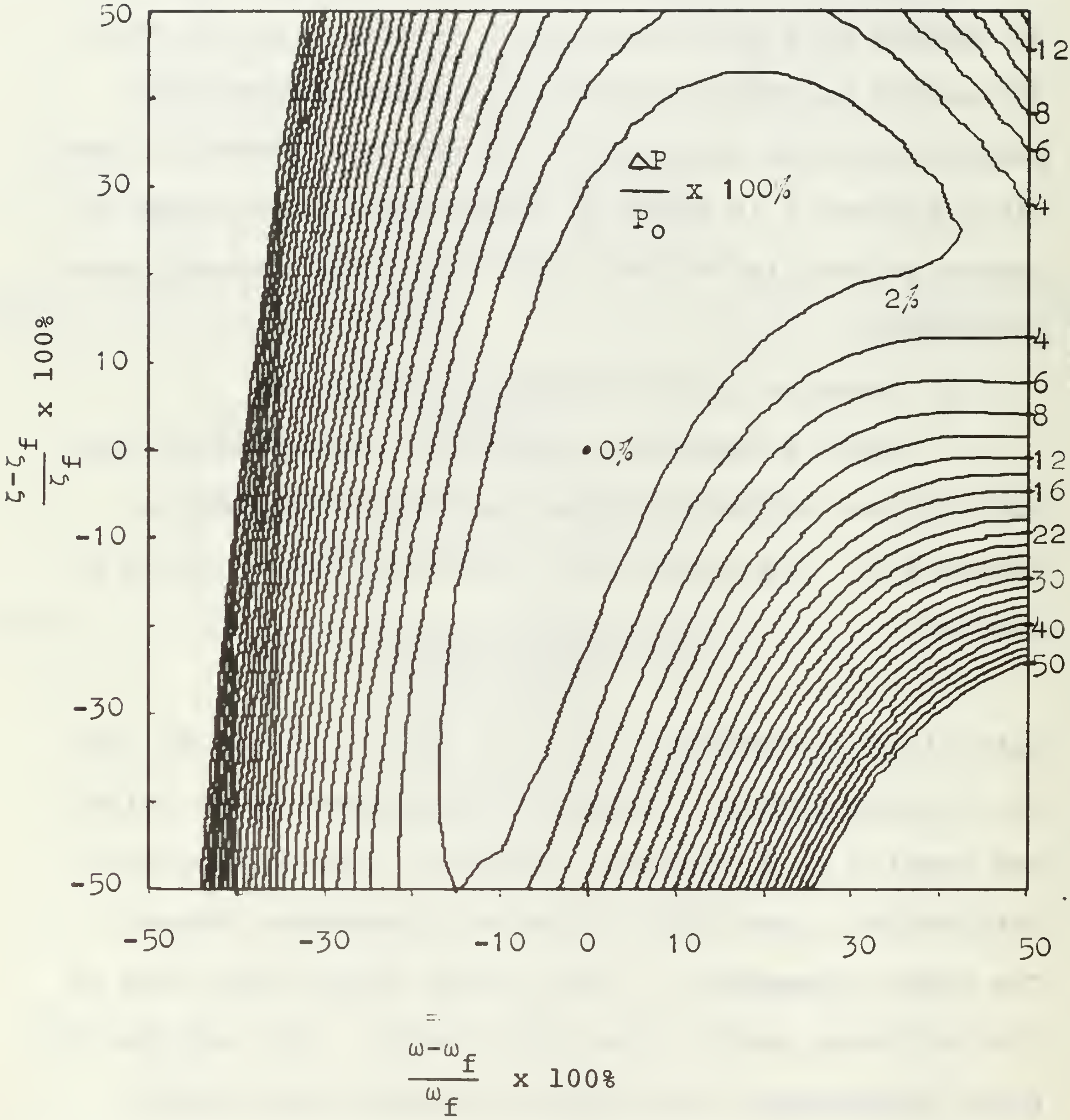


Fig. 4-2 Degradation vs Parameter Errors
(2-parameter model)

are then contoured by linear interpolation into the $\zeta$-$\omega$ plane. Figure 4-2 shows the resulting contour map. The contours are marked as a percentage degradation from the optimal trace as a function of the percentage error in the filter parameters. Such a graph could assist not only in the type of decision mentioned earlier, but also in a determination of which direction of error is more costly by considering the "gradient" of the surface in the various directions in the parameter plane.

c. MODEL ORDER REDUCED BY ONE

The case in which a third order system with a complex conjugate pair of poles and a remote real pole is to be filtered by a second order model was also considered as a numerical example. The erroneous filter model was based on the plant transfer function

$$\frac{X(S)}{U(S)} = \frac{2}{s^2 + 2s + 2} \qquad (4\text{-}2)$$

While optimum filter used the model

$$\frac{X(S)}{U(S)} = \frac{2a}{(s^2 + 2s + 2)(s + a)} \qquad (4\text{-}3)$$

where a was allowed to vary from 50 to 0.5 in increments of 0.5. Both models are type 0 with the same complex poles, since this is considered to be a case of deliberate misidentification.

The recursive matrix equations were made compatible by the addition of a row and column of zeros in the state transition matrix for the filter and used as before.

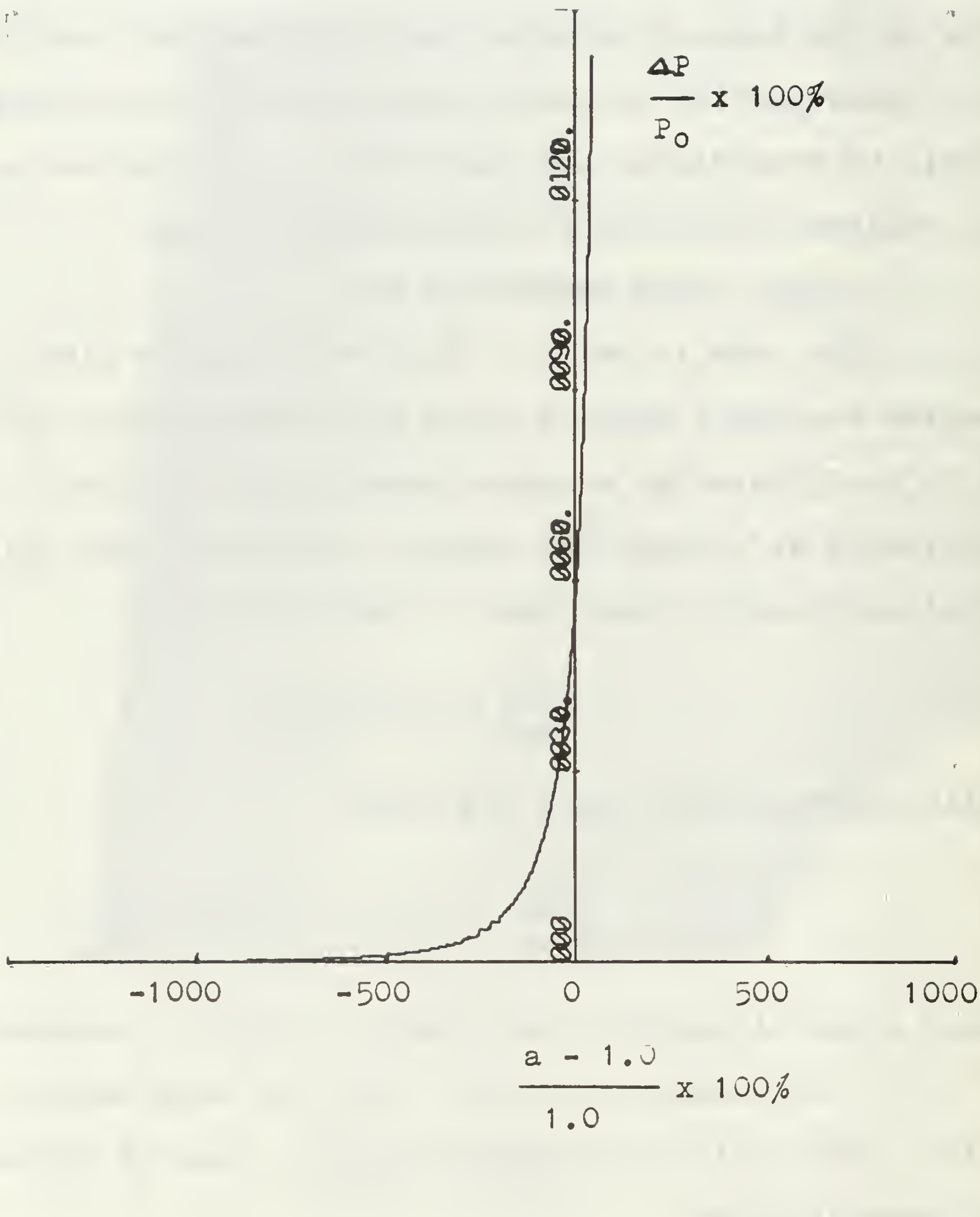


Fig. 4-3 Degradation vs Parameter Errors
(3 parameter model, 1 incorrect)

Another minor difference is that only the upper two elements of the diagonal were used when considering the trace of the covariance matrix of estimation error, since only two of the three plant states were estimated. The parameter becomes the value of the real part for the remote pole and the expected result is a monotonic increasing degradation as the pole location becomes less remote. Figure 4-3 shows a typical graph of this result.

d. MODEL ORDER REDUCED BY TWO

A numerical example similar to c above was simulated in which a fourth order type 0 system with two complex conjugate pole pairs was reduced to a second order filter type 0 model with the dominant pole pair identified exactly. The sensitivity parameters were taken as the damping factor $\zeta$ and natural frequency $\omega$ associated with the remote complex pole pair. The pole locations for the plant were taken to be representative of the short period and phugoid oscillatory modes in the linearized model of an aircraft over a limited flight regime. The interpretation would be to find the degradation in estimation of altitude and altitude rate which results from ignoring the short period vertical oscillations of the airframe produced by elevator perturbations and air gusts. The results are shown in Figure 4-4. The numerical values used for the accurate model were

$$\omega_1 = 1.15,\ \zeta_1 = .35,\ \omega_2 = .073,\ \zeta_2 = .035 \qquad (4\text{-}4)$$

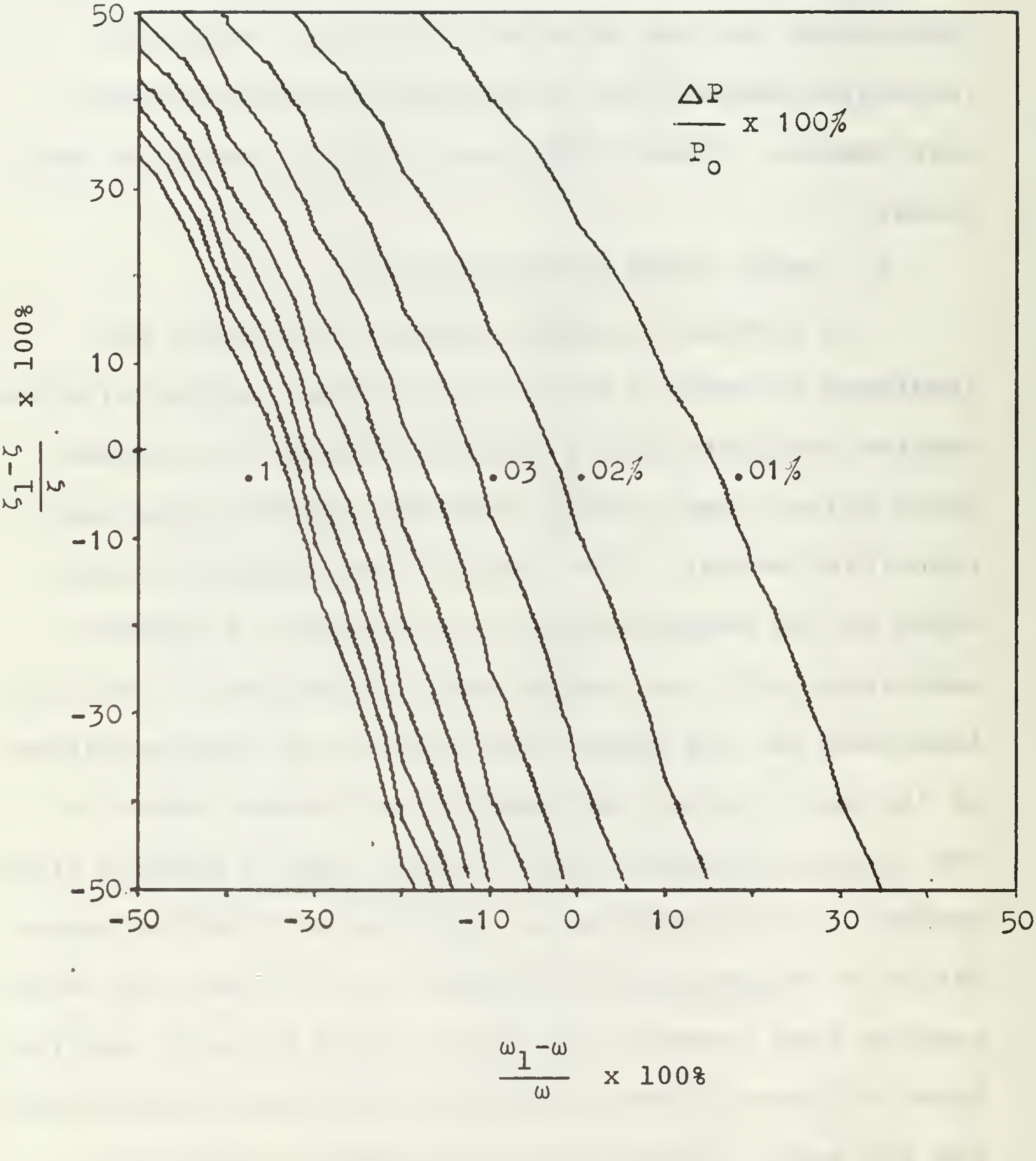


Fig. 4-4 Degradation vs Parameter Errors
(4 parameter model, 2 incorrect)

while the erroneous model was taken as

$$\omega = .073, \; \zeta = .035 \tag{4-5}$$

The parameter $\zeta_1$ was allowed to vary from .175 to .525, with values of $\omega_1$ from .575 to 1.725.

CHAPTER 5

## CONCLUSIONS

The principal result of this investigation has been the derivation of an algorithm to replace the Kalman filter gain calculation when errors in the model of the dynamics of an observed system are known to exist. This algorithm can be used in two ways: either as a means for producing optimal estimates in a low order filter, or to determine the cost of parameter mis-identification in terms of estimation accuracy for some specific system. In the first application, the reduction in computation time associated with low order filtering is partially negated by the requirement for making two additional calculations at each iteration. Therefore such an application probably would become profitable only if the system model order can be reduced by two or more in the filter. It is felt that the use of equations 3-18, 3-15, 3-20 and 3-21 with various suboptimal gain sequences could assist in numerous design studies. An example would be the study of filter performance degradation where a single filter model is to be used with many plants, each having slightly different parameters. Another example would be application of a Kalman filter scheme to a system with parameters which vary slowly with time.

Development of the recursive expressions for calculating the actual covariance of estimation error, together with the computer simulation to test their utility, has

revealed some interesting sidelights. Perhaps the most significant of these is the difficulty in obtaining a sensitivity function in the usual sense for other than the scalar cases. The second order, two-parameter case yields a set of four simultaneous non-linear matrix equations from which the partial derivatives must be produced. Thus, the sensitivity function approach was abandoned in favor of the recursive solution of actual degradation.

Another interesting result was the fact that, in the particular numerical examples used in Chapter 4 the filter performance degradation was not nearly as great as the authors had anticipated. The two-parameter degradation contours shown in Figure 4-2 describe a bowl-shaped surface in the "parameter error plane" as would be expected. However, the surface has a relatively flat bottom and allows considerable parameter error in certain directions without exceeding a one per cent degradation in performance. In view of the analytical results of the scalar examples in Chapter 3, the gradient of this surface near its minimum is considered to depend heavily on the values chosen for $\Omega$ and R. It is known that the ratio $\Omega/R$ greatly affects variance reduction in most tracking filters. The higher this value, the greater will be the variance reduction. The numerical ratio used in all examples was 100 which is probably optimistic. A study

of the effect of this ratio on the gradient of the surface of figure 4-2 might verify the foregoing remarks.

From the examples of a low-order filter model, it appears that the idea of second order dominance for more complicated systems may have promise in certain digital filter applications. Although each specific application requires a simulation such as those in Chapter 4, much of the guesswork associated with exactly what constitutes second order dominance can be eliminated, once the simulation is performed. This subject might warrant further investigation to learn just how "remote" higher order system poles must be, how the values $\Omega$, R, and $\Omega$/R affect estimation accuracy, etc.

A related effect of erroneous filter models noted in this investigation was a significant increase in the number of iterations required to achieve "steady state" in the calculated covariance as model errors increased. The filter "settling time" naturally depends heavily on initialization of $\underline{\hat{x}}(0/0)$ and P(0/0), but dependence on errors in the plant model can further aggravate the situation. Filter settling time or "lock-on" can be very critical in certain applications such as fire control systems. This is another area which could be explored further.

While the main objectives of this investigation have been realized, the Kalman filter is far from a dead

issue. On the contrary, completion of this work has served to open several new questions which can lead to successful application of the theoretical concepts embodied in optimal state estimation.

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**13. ABSTRACT**

This investigation is concerned with the effects of employing a Kalman filter to estimate the states in a system for which the mathematical model is inaccurate. Consideration is given to both intentional and unintentional mis-identification of parameters in the assumed plant dynamics. An algorithm consisting of four matrix equations is derived which yields the actual covariance of estimation error when errors in the assumed model are known. Depending upon the gain sequence used, the derived equations can be used to either 1) produce optimal estimates when errors are deliberate or 2) aid in the determination of mis-identification costs in terms of filter performance degradation if the relative accuracy of parameter identification is known.

Analytic examples of scalar cases are included, as well as computer simulations for specific higher order systems, including the employment of a second order filter model with a fourth order plant.

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